Tropical Ideals, genera of tropicalization of curves and the minimum finishing time of projective networks

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~Tropical Ideals~

●Background

- For a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$, the tropical variety $\mathbf{V}(f) \subset \mathbb{T}^n$ is the support of a finite polyhedral complex in $\mathbb{T}^n$.

- If we define the tropical variety $\mathbf{V}(I)$ defined by an ideal I in $\mathbb{T}[x_1, \dots, x_n]$ as
$$\mathbf{V}(I) = \bigcap_{f \in I} \mathbf{V}(f),$$
then the variety $\mathbf{V}(I)$ is **not** always the support of a finite polyhedral complex (Example 5.14 in [1]).

- Maclagan and Rincón defined **tropical ideals** in [1]. The tropical variety defined by a tropical ideal is the support of a finite polyhedral complex.

●Motivations

It is difficult to treat tropical ideals like classical one because they are not closed under the addition, multiplication or intersection.
(We cannot even "generate" a tropical ideal from an arbitrary set of tropical polynomials.)

We want to make another "tropical ideals" such that
(a) The tropical variety defined by any of them is the support of a finite polyhedral complex,
(b) They are closed under the addition, multiplication and intersection.

●Notation

- $\mathbb{T}$: **Tropical semifield**.
  i.e. the semifield $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where
  $a \oplus b \coloneqq \max\{a, b\}$ (addition)
  $a \odot b \coloneqq a + b$ (multiplication).

- $\mathbb{T}[x_1, \dots, x_n]$: **Tropical polynomial semiring**

[1] D. Maclagan and F. Rincón, Tropical ideals, Discrete Mathematics & Theoretical Computer Science proc. **BC** (2016), pp. 803-814.

●Definition

The **tropical polynomial function semiring** is the quotient semiring $\mathbb{T}[x_1, \dots, x_n]/\sim$, where $\sim$ is defined as
$$f \sim g \Leftrightarrow f(\boldsymbol{a}) = g(\boldsymbol{a}) \quad \forall \boldsymbol{a} \in \mathbb{R}^n.$$

The **tropical variety** $\mathbf{V}(f)$ defined by a tropical polynomial $f \in \mathbb{T}[x_1, \dots, x_n]$ is
$$\mathbf{V}(f) = \left\{ \mathbf{a} \in \mathbb{T}^n \,\middle|\, \begin{array}{l} \text{The maximum of } f(\mathbf{a}) \text{ is attained at} \\ \text{least twice or } f(\mathbf{a}) = -\infty. \end{array} \right\}.$$

We may define the tropical variety $\mathbf{V}(\varphi)$ just for a tropical polynomial function φ.

We denote by $[f]_{\mathbf{x^u}}$ the coefficient of the monomial $\mathbf{x^u}$ in a tropical polynomial f.

The **maximum representation** $\varphi^{\max}$ of a tropical polynomial function φ is the representation f such that
$$[f]_{\mathbf{x^u}} \geq [g]_{\mathbf{x^u}}$$
for any representation g of φ and any monomial $\mathbf{x^u}$.

Each tropical polynomial function φ has a unique maximum representation.

An ideal I in $\mathbb{T}[x_1, \dots, x_n]/\sim$ is a **tropical ideal** if for any $\varphi, \psi \in I$ and any monomial $\mathbf{x^u}$ with $[\varphi^{\max}]_{\mathbf{x^u}} = [\psi^{\max}]_{\mathbf{x^u}} \neq -\infty$, there is a tropical polynomial h such that,
(1) the class of h is in I,
(2) $[h]_{\mathbf{x^u}} = -\infty$,
(3) $[h]_{\mathbf{x^v}} \leq [\varphi^{\max}]_{\mathbf{x^v}} \oplus [\psi^{\max}]_{\mathbf{x^v}}$ for all $\mathbf{v}$, and
(4) $[h]_{\mathbf{x^v}} = [\varphi^{\max}]_{\mathbf{x^v}} \oplus [\psi^{\max}]_{\mathbf{x^v}}$ if $[\varphi^{\max}]_{\mathbf{x^v}} \neq [\psi^{\max}]_{\mathbf{x^v}}$.

· **Maclagan and Rincón's definition of tropical ideals**

An ideal I in $\mathbb{T}[x_1, \dots, x_n]$ is a **tropical ideal** if for any $f, g \in I$ and any monomial $\mathbf{x^u}$ with $[f]_{\mathbf{x^u}} = [g]_{\mathbf{x^u}} \neq -\infty$, there is a tropical polynomial h such that,
(1) $h \in I$,
(2) $[h]_{\mathbf{x^u}} = -\infty$,
(3) $[h]_{\mathbf{x^v}} \leq [f]_{\mathbf{x^v}} \oplus [g]_{\mathbf{x^v}}$ for all $\mathbf{v}$, and
(4) $[h]_{\mathbf{x^v}} = [f]_{\mathbf{x^v}} \oplus [g]_{\mathbf{x^v}}$ if $[f]_{\mathbf{x^v}} \neq [g]_{\mathbf{x^v}}$.

●Main Results

Theorem 1. (principal ⇒ tropical)
For any tropical polynomial function $\varphi \in \mathbb{T}[x]/\sim$, the set $\varphi \odot \mathbb{T}[x]/\sim \coloneqq \{\varphi \odot \psi \mid \psi \in \mathbb{T}[x]/\sim\}$ is a tropical ideal in $\mathbb{T}[x]/\sim$.

Theorem 2. (like PID)
Every tropical ideal in $\mathbb{T}[x]/\sim$ is of the form $\varphi \odot \mathbb{T}[x]/\sim$ for some $\varphi \in \mathbb{T}[x]/\sim$.

Corollary 3. (intersect & generate)
Tropical ideals in $\mathbb{T}[x]/\sim$ are closed under the intersection. Hence for any set S of tropical polynomial functions, there is the minimum tropical ideal including S.

~ Genera of tropicalization of curves ~

●Motivations

Let C be a projective curve over a valued field K.

Q. If K has "many" valuations, can we find the tropicalization whose genus is equal to the one of C by varying valuations?

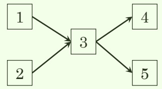
●Theorem

K : an algebraic function field
C : an elliptic curve on K with nonconstant j-invariant.

Theorem.
In this situation,
$\exists K'$: a finite extension of K,
$\exists C' \hookrightarrow \mathbb{P}^2_{K'}$: an embedding, where C' is the scalar extension of C to K', and
$\exists v$: a valuation on K'
such that the tropicalization of C' via v has the genus 1.

~The minimum finishing time of projective networks~

A **project network** consists of some activities, where each activity can be started after all the preceding activities have finished.



In the above project network, let t_i be the time to complete the activity. Then the **minimum finishing time** of this project network is

$\max\{t_1 + t_3 + t_4,\ t_1 + t_3 + t_5,\ t_2 + t_3 + t_4,\ t_2 + t_3 + t_5\}$
$= t_1 t_3 t_4 \oplus t_1 t_3 t_5 \oplus t_2 t_3 t_4 \oplus t_2 t_3 t_5$ (in tropical notation),

which is a tropical polynomial of t_i's.

Q. What kind of tropical polynomials can be realized as the minimum finishing time of a project network?

Definition
A ***P*-polynomial** is a tropical polynomial $f(t)$ such that
(1) the degree on each variable is exactly one,
(2) the coefficient of each term is a unity,
(3) no term is divisible by any other terms.

Definition
A P-polynomial $f(t) = f(t_1, \dots, t_n)$ has **term extendability** if, for any subset $I \subset [n]$ such that
$\forall i, j \in I$, there is a term of $f(t)$ divisible by $t_i t_j$,
there is a term of $f(t)$ divisible by $\prod_{i \in I} t_i$.

Theorem 1.
There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \dots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$.

We denote by $\mathrm{TG}(f)$ the simple graph corresponding to $f(t)$.

Theorem 2.
Let $f(t)$ be a P-polynomial of degree d having term extendability. Then $f(t)$ is realizable if and only if there is a vertex coloring of $\mathrm{TG}(f)$ with the color set $\{1, \dots, d\}$ such that if v_1, v_2, v_3 is colored by c_1, c_2, c_3 with $c_1 < c_2 < c_3$ and $\{v_1, v_2\}$, $\{v_2, v_3\}$ are adjacent respectively, then $\{v_1, v_3\}$ is also adjacent.

[1] Ito, T., A characterization for tropical polynomials being the minimum finishing time of project networks, (2017) Hokkaido Mathmatical Journal, in press.

[2] Kobayashi, M. and Odagiri, S., Tropical geometry of PERT, Journal of Math-for-Industry, Vol. 5 (2013B-8), pp. 145-149